XII. *On the Contact of Surfaces.* *By* WILLIAM SPOTTISWOODE, *M.A.*, *Treas. R.S.*

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IN a paper published in the Philosophical Transactions (1870, p. 289) I have considered the contact, at a point P, of two curves which are coplanar sections of two surfaces (U, V), and have examined somewhat in detail the case where one of the curves, viz. the section of V, is a conic. In the method there employed, the condition that the point P should be sextactic, involved the azimuth of the plane of section measured about an axis passing through P; and consequently, regarded as an equation in the azimuth, it showed that the point would be sextactic for certain definite sections. It does not, however, follow, if conics having six-pointic contact with the surface U be drawn in the planes so determined, that a single quadric surface can be made to pass through them all. In fact it will be shown in the sequel that when this is possible, the quadric in general degenerates into a pair of planes. The investigation therefore of the memoir above quoted was not directly concerned with the contact of surfaces, although it may be regarded as dealing with a problem intermediate to the contact of plane curves and that of surfaces.

In the present investigation I have considered a point P common to the two surfaces U and V, an axis drawn arbitrarily through P, and a plane of section passing through the axis and capable of revolution about it. Proceeding as in the former memoir, and forming the equations for contact of various degrees, and finally rendering them independent of the azimuth, we obtain the conditions for contact for all positions of the cutting plane about the axis. Such contact is called circumaxal; and in particular it is called uniaxal, biaxal, &c. according as it subsists for one, two, &c. axes. If it holds good for all axes through the point, it is called superficial contact.

It would at first sight seem that there should be a similar theory as to the number of axes about which contact must subsist in order that it may subsist about all axes, or be superficial. It is, indeed, found that if two-pointic contact be biaxal, or if three-pointic be triaxal, &c., the contact will be superficial. But this would prove too much, as it would give four conditions instead of two for two-pointic, six conditions instead of three for three-pointic, &c. superficial contact; and, in fact, it turns out that there are always in two-pointic contact one, in three-pointic two, &c. axes (viz. the tangents to the branches of the curve of contact through the point) about which the contact is circumaxal *per se*, so that the theory in one sense disappears. But as it at first had a semblance of existence, it may still be worth while to have laid its ghost.

At the conclusion of § 3 it is shown that the method of plane sections may, in the cases possessing most interest and importance, be replaced by the more general method of curved sections.

In the concluding section a few general considerations are given relating to the determination of surfaces having superficial contact of various degrees with given surfaces; and at the same time it is indicated how very much the general theory is affected by the particular circumstances of each case. The question of a quadric having four-pointic superficial contact with a given surface is considered more in detail; and it is shown how in general such a quadric degenerates into the tangent plane taken twice. To this there is apparently an exceptional case, the condition for which is given and reduced to a comparatively simple form; but I must admit to having so left it, in the hope of giving a fuller discussion of it on a future occasion.

The subject of three-pointic superficial contact was considered by DUPIN, 'Développements de Géométrie,' p. 12; and, as I have learnt since the memoir was written, a general theorem connecting superficial contact and contact along various branches of the curve of intersection of two surfaces (substantially the same as that given in the text) was enunciated by M. MOUTARD*.

§ 1. *Preliminary Formulæ and Transformations.*

Let $U=0$, $V=0$ be the equations of the two surfaces whose contact is the subject of investigation. Let their degrees be m and n respectively; and let, as usual,

$$\left.\begin{array}{llll} \partial_x U=u\,, & \partial_y U=v\,, & \partial_z U=w\,, & \partial_t U=k\,, \\ \partial_x^2 U=u_1, & \partial_y^2 U=v_1, & \partial_z^2 U=w_1, & \partial_t^2 U=k_1, \\ \partial_y\partial_z U=u', & \partial_z\partial_x U=v'\,, & \partial_x\partial_y U=w', & \\ \partial_x\partial_t U=l'\,, & \partial_y\partial_t U=m', & \partial_z\partial_t U=n'. & \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also let $\bar{u}, \bar{v}, \ldots \bar{u}_1, \bar{v}_1, \ldots \bar{u}', \ldots \bar{l}', \ldots$ represent the corresponding differential coefficients of V.

Further, $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$ being arbitrary quantities, let

$$\left.\begin{array}{c} \alpha x+\beta y+\gamma z+\delta t=\varpi\,, \\ \alpha' x+\beta' y+\gamma' z+\delta' t=\varpi', \\ \alpha'\varpi-\alpha\varpi'=\mathrm{A}, \\ \beta'\varpi-\beta\varpi'=\mathrm{B}, \\ \gamma'\varpi-\gamma\varpi'=\mathrm{C}, \\ \delta'\varpi-\delta\varpi'=\mathrm{D}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also, forming the determinants in the usual order, let

$$a, b, c, f, g, h=\left\|\begin{array}{cccc} \alpha, & \beta, & \gamma, & \delta, \\ \alpha', & \beta', & \gamma', & \delta', \end{array}\right\| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

* Poncelet, 'Applications d'Analyse à la Géométrie,' 1864, tom. ii. p. 363.

viz. a, b, c, f, g, h are the six coordinates of the line of intersection of the planes ϖ, ϖ', or *the axis* as it may be termed; and $A=0, B=0, C=0, D=0, Ax+By+Cz+Dt=0$ represent five planes passing through the axis, the azimuth of the last plane about the axis being determined by the quantity $\varpi' : \varpi$.

This being so, the quantities $a, b, .. A, B, ..$ will be found to satisfy the following relations:—

$$\left.\begin{array}{l} Bh-Cg+Da=0, \\ Cf-Ah+Db=0, \\ Ag-Bf+Dc=0, \\ Aa+Bb+Cc=0, \\ af+bg+ch=0; \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (4)$$

while for every point of the plane $Ax+By+Cz+Dt=0$ we shall have also

$$\left.\begin{array}{r} bz-cy-ft=A, \\ cx-az-gt=B, \\ ay-bx-ht=C, \\ fx+gy+hz=D, \\ Ax+By+Cz+Dt=0. \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (4a)$$

Again, let

$$\square V=\left|\begin{array}{cccc} \alpha, & \beta, & \gamma, & \delta, \\ \alpha', & \beta', & \gamma', & \delta', \\ u, & v, & w, & k, \\ \partial_x, & \partial_y, & \partial_z, & \partial_t, \end{array}\right| V. \quad \ldots\ldots\ldots \quad (5)$$

Then if we write

$$x(\text{column } 1)+y(\text{column } 2)+z(\text{column } 3)+t(\text{column } 4)$$

for each of the columns in succession of the expression for $\square V$, it will be found that the following transformation may be effected, viz.

$$x\square V,\ y\square V,\ z\square V,\ t\square V=\left\|\begin{array}{cccc} A, & B, & C, & D, \\ u, & v, & w, & k, \\ \partial_x, & \partial_y, & \partial_z, & \partial_t, \end{array}\right\| V. \quad \ldots\ldots \quad (6)$$

Again, let

$$\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6=\left\|\begin{array}{cccc} u, & v, & w, & k, \\ \partial_x, & \partial_y, & \partial_z, & \partial_t, \end{array}\right\|. \quad \ldots\ldots\ldots \quad (7)$$

then it will be found that

$$\left.\begin{aligned}\square V&=(a\delta_4+b\delta_5+c\delta_6+f\delta_1+g\delta_2+h\delta_3)V,\\ x\square V&=(B\delta_6-C\delta_5+D\delta_1)V=(\varpi'\square_1-\varpi\square'_1)V,\\ y\square V&=(C\delta_4-A\delta_6+D\delta_2)V=(\varpi'\square_2-\varpi\square'_2)V,\\ z\square V&=(A\delta_5-B\delta_4+D\delta_3)V=(\varpi'\square_3-\varpi\square'_3)V,\\ t\square V&=(A\delta_1+B\delta_2+C\delta_3)V=(\varpi'\square_4-\varpi\square'_4)V,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where

$$\left.\begin{aligned}\square_1&=\beta\delta_6-\gamma\delta_5+\delta\delta_1, & \square'_1&=\beta'\delta_6-\gamma'\delta_5+\delta'\delta_1,\\ \square_2&=\gamma\delta_4-\alpha\delta_6+\delta\delta_2, & \square'_2&=\gamma'\delta_4-\alpha'\delta_6+\delta'\delta_2,\\ \square_3&=\alpha\delta_5-\beta\delta_4+\delta\delta_3, & \square'_3&=\alpha'\delta_5-\beta'\delta_4+\delta'\delta_3,\\ \square_4&=\alpha\delta_1+\beta\delta_2+\gamma\delta_3, & \square'_4&=\alpha'\delta_1+\beta'\delta_2+\gamma'\delta_3.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Lastly, the operators δ_1, δ_2, .. are subject to the following identical relations, viz. :—

$$\left.\begin{aligned}v\delta_6-w\delta_5+k\delta_1&=0,\\ w\delta_4-u\delta_6+k\delta_2&=0,\\ u\delta_5-v\delta_4+k\delta_3&=0,\\ u\delta_1+v\delta_2+w\delta_3&=0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

by means of which we may always eliminate one of the three operators entering into each of the expressions (8). In fact, the following values would express the result of such an elimination:—

$$\left.\begin{aligned}k(B\delta_6-C\delta_5+D\delta_1)&=(Bk-Dv)\delta_6-(Ck-Dw)\delta_5,\\ k(C\delta_4-A\delta_6+D\delta_2)&=(Ck-Dw)\delta_4-(Ak-Du)\delta_6,\\ k(A\delta_5-B\delta_4+D\delta_3)&=(Ak-Du)\delta_5-(Bk-Dv)\delta_4;\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

so that in the case where $\square V=0$, we should obtain

$$\delta_1V:\delta_2V:\delta_3V:\delta_4V:\delta_5V:\delta_6V=\left\Vert\begin{matrix}A, & B, & C, & D,\\ u, & v, & w, & k.\end{matrix}\right\Vert \quad . \quad . \quad . \quad . \quad (12)$$

There is one other mode of transformation which, on account of its utility, may properly find a place here. If U_0, U_1, be the same linear functions of α, β, γ, δ, and α', β', γ', δ', respectively, say

$$\left.\begin{aligned}U_0&=(\alpha, \ \beta, \ \gamma, \ \delta),\\ U_1&=(\alpha', \ \beta', \ \gamma', \ \delta'),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

then it will be found that

$$\left.\begin{aligned}(U_0, U_1)(\alpha', -\alpha)&=(\ \ .\ \ \ \ c, \ -b, \ f),\\ (U_0, U_1)(\beta', -\beta)&=(-c, \ \ .\ \ \ \ a, \ g),\\ (U_0, U_1)(\gamma', -\gamma)&=(\ b, \ -a, \ \ .\ \ \ \ h),\\ (U_0, U_1)(\delta', -\delta)&=(\ f, \ g, \ h, \ \ \ ).\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Similarly, if U_0, U_1, U_2 be the same functions, the first quadratic in $\alpha, \beta, \gamma, \delta$; the second lineo-linear in $\alpha, \beta, \gamma, \delta$; $\alpha', \beta', \gamma', \delta'$; the third quadratic in $\alpha', \beta', \gamma', \delta'$, say,

$$\left.\begin{aligned} U_0 &= (\alpha, \beta, \gamma, \delta)^2, \\ U_1 &= (\alpha, \beta, \gamma, \delta)(\alpha', \beta', \gamma', \delta'), \\ U_2 &= (\alpha', \beta', \gamma', \delta'), \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (15)$$

then it will be found that

$$\left.\begin{aligned} (U_0, U_1, U_2), (\alpha', -\alpha)^2 &= (\,.\, c, -b, f)^2, \\ (U_0, U_1, U_2), (\beta', -\beta)^2 &= (-c, \,.\, a, g)^2, \\ \vdots \qquad \vdots \qquad & \quad \vdots \\ (U_0, U_1, U_2)(\beta', -\beta)(\gamma', -\gamma) &= (-c, \,.\, a, g)(b, -a, \,.\, h), \\ \vdots \qquad \vdots \qquad & \quad \vdots \\ (U_0, U_1, U_2)(\alpha', -\alpha)(\delta', -\delta) &= (\,.\, c, -b, f)(f, g, h, .). \\ \vdots \qquad \vdots \qquad & \quad \vdots \end{aligned}\right\} \quad \ldots \quad (16)$$

And a similar process is also obviously applicable to functions of higher degrees.

§ 2. *Conditions of Contact.*

In the memoir "On the Contact of Conics with Surfaces," above quoted, it was shown that the conditions for a 1, 2, .. pointic contact at a point P of the curves of section of the surfaces U, V, made by a plane $Ax+By+Cz+Dt=0$, may be expressed as follows:

$$V=0, \quad \square V=0, \quad \square^2 V=0\ldots, \quad \ldots \ldots \quad (17)$$

The cutting plane, say the plane (A, B, C, D), was supposed to pass through the point (x, y, z, t), say the point P, to be capable of revolving about the axis whose six coordinates are (a, b, c, f, g, h), and to have an azimuth which, measured about the axis, is determined by the quantity $\varpi' : \varpi$.

The equations (17) may be supposed to be expressed in any of the forms to which they were reduced in § 1. Taking, for instance, the form (8), and dropping for the present the suffixes, so that $\square$, $\square'$ shall be understood to represent any pair of the operators $\square_1, \ldots \square'_1, \ldots$, the equations (17) may be written thus:

$$V=0, \ (\varpi'\square-\varpi\square')V=0, \quad (\varpi'\square-\varpi\square')^2V=0\ldots, \quad \ldots \ldots \quad (18)$$

In the expansion of these expressions there will occur combinations such as $\delta_i\delta_j V$, where i and j represent any of the numbers 1, 2, 3, 4; and when i and j are different, the compound operation $\delta_j\delta_i$ is not in general the same as $\delta_i\delta_j$. But in the case where $\delta_i V=0$, $\delta_j V=0$, it will be found on actual trial of the special forms that $\delta_j\delta_i V=\delta_i\delta_j V$. The same thing may be also proved by the following considerations, which are applicable to all the forms of δ_i and δ_j. In the first place δ_i is symmetrical, except as regards algebraical sign, in respect of the first differential coefficients of U and V; so that if δ_i be the expression obtained by replacing the differential coefficients of U by those of $-V$ in δ_i, we may write $\delta_i V=\bar{\delta}_i U$. Secondly, the result of any combined operation such as $\delta_j\delta_i U$ consists of two parts, say $\delta'_j\delta_i V$ (or $\delta'_j\delta'_i V$), where the accented operators affect V only, and $\delta''_j\delta_i V=\delta''_j\bar{\delta}_i U$ (or $\delta''_j\bar{\delta}''_i U$), where the doubly accented operators affect U only. But the conditions $\delta_i V=0$, $\delta_j V=0$ give, as is well known, $\partial_x V=\theta u$, $\partial_y V=\theta v$, .., where θ is indeterminate; so that if, after the operations $\delta''_j\bar{\delta}''_i$ have been performed on U, we replace $\partial_x V$ by θu, $\partial_y V$ by θv, we shall have an expression identical with that obtained by evaluating $\delta''_i\bar{\delta}''_j U$. In other words,

$$\text{when } \delta_i V=0,\quad \delta_j V=0,\quad \text{then } \delta_j\delta_i V=\delta_i\delta_j V. \qquad (19)$$

Similarly, if, in addition to $\delta_i V=0$, $\delta_j V=0$, we have

$$\delta_i^2 V=0,\ \delta_j\delta_i V=0,\ \delta_j^2 V=0, \qquad (20)$$

then

$$\left.\begin{array}{ll} \because\ \delta_i^2\,V=0,\quad \delta_j\delta_i V=0, & \therefore\ \delta_i\delta_j\delta_i V=\delta_j\delta_i^2 V, \\ \because\ \delta_j^2\,V=0,\quad \delta_i\delta_j V=0, & \therefore\ \delta_j\delta_i\delta_j V=\delta_i\delta_j^2 V, \\ \because\ \delta_i\delta_j V=\delta_j\delta_i V=0, & \therefore\ \delta_i^2\delta_j\delta_j V=\delta_j\delta_i^2 V. \end{array}\right\} \qquad (21)$$

The equations (18) may be regarded as equations involving an unknown quantity $\varpi':\varpi$, which determines the section along which there is a contact of a given degree. Thus, in order that the surfaces may have a p-pointic contact at the point P along some section through the axis, we must have

$$(\varpi'\square-\varpi\square')^{p-1}V=0. \qquad (22)$$

But, inasmuch as whenever there is a p-pointic contact along any section there must also be a $p-1$, $p-2$, .. pointic contact, it follows that, in addition to the condition above written, we must also have

$$(\varpi'\square-\varpi\square')^{p-2}V=0,\quad (\varpi'\square-\varpi\square')^{p-3}V=0\ ..; \qquad (23)$$

and the conditions for the existence of a p-pointic contact at the point P along some section through the axis will be expressed by eliminating $\varpi':\varpi$ from the equations (22), (23), taken two and two together.

Thus the condition for a three-pointic would be

$$(\square' V)^2\square^2 V-\square' V\square V(\square\square'+\square'\square)V+(\square V)^2\square'^2 V=0, \qquad (24)$$

or

$$(\square^2 V,\ (\square\square'+\square'\square)V,\ \square'^2 V)(\square' V,\ -\square V)^2=0. \qquad (25)$$

Similarly, for a four-pointic contact we should have, in addition to (25), the following:

$$(\square^3 V,\ (\square'\square^2+\square\square'\square+\square^2\square')V,\ (\square'^2\square+\square'\square\square'+\square\square'^2)V,\ \square'^3 V)(\square' V,\ -\square V)^3=0, \qquad (26)$$

and so on, for higher degrees, the azimuth of the plane section being always determined by the value of $\square' V:\square V$ at the point P. And it may be observed that, under the conditions supposed, there will in general be only one plane section along which the contact will subsist.

If the surfaces touch at the point P, the equation $(\varpi'\Box-\varpi\Box')V=0$ is satisfied identically, and there will in general be two directions, determined by the equation

$$(\varpi'\Box-\varpi\Box')^2V=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

along which there will be three-pointic contact, as will be further noticed in the next section. The condition for a four-pointic contact will then be obtained by eliminating $\varpi':\varpi$ from the equations (27) and (26). These considerations may readily be extended to higher degrees.

It will perhaps be worth while before proceeding further to evaluate the expression (24); viz. the condition to be fulfilled in order that when two surfaces meet at a point P but do not touch, the curves of section made by some plane passing through a given axis through P shall have three-pointic contact. With a view to this, let us take the following form for $\Box$, viz.

$$\Box=\alpha\delta_1+\beta\delta_2+\gamma\delta_3, \quad \Box'=\alpha'\delta_1+\beta'\delta_2+\gamma'\delta_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Then the expression to be developed will be

$$\begin{aligned}&(\alpha'\delta_1V+..)^2(\alpha^2\delta_1^2+\beta^2\delta_2^2+..+2\beta\gamma(\delta_1\delta_2+\delta_2\delta_1)+..)V\\ &-(\alpha\delta_1V+..)(\alpha'\delta_1V+..)(2\alpha\alpha'\delta_1^2+2\beta\beta'\delta_2^2+..(\beta\gamma'+\beta'\gamma)(\delta_1\delta_2+\delta_2\delta_1)+..)V\\ &+(\alpha\delta_1V+..)^2(\alpha'^2\delta_1^2+\beta'^2\delta_2^2+..+2\beta'\gamma'(\delta_1\delta_2+\delta_2\delta_1)+..)V.\end{aligned}$$

In this it will be found that the coefficients of $\alpha'^2\alpha^2$, $\beta'^2\beta^2$, .. vanish, and that the coefficients of $\alpha^2\beta'^2$, $-2\alpha\alpha'\beta\beta'$, $\alpha'^2\beta^2$ are all the same, viz.

$$(\delta_1V)^2\delta_2^2V-\delta_1V\delta_2V(\delta_1\delta_2+\delta_2\delta_1)V+(\delta_2V)^2\delta_1^2V.$$

Hence the whole expression may be reduced to the form

$$\{a^2(\delta_2V\delta_3-\delta_3V\delta_2)^2+..+2bc(\delta_3V\delta_1-\delta_1V\delta_3)(\delta_1V\delta_2-\delta_2V\delta_1)+..\}V, \quad . \quad . \quad (29)$$

it being understood that the operations δ_1, δ_2, δ_3 do not affect the quantities δ_1V, δ_2V, δ_3V so far as they appear explicitly in the above expression. In order to calculate the coefficients of the powers and products of a, b, c, we have

$$\left.\begin{aligned}&\delta_1V=v\bar{w}-w\bar{v}, \quad \delta_2V=w\bar{u}-u\bar{w}, \quad \delta_3V=u\bar{v}-v\bar{u},\\ &\delta_1u=vv'-ww', \quad \delta_2u=wu_1-uv', \quad \delta_3u=uw'-vu_1,\\ &\delta_1v=vu'-wv_1, \quad \delta_2v=ww'-uu', \quad \delta_3v=uv_1-vw',\\ &\delta_1w=vw_1-wu', \quad \delta_2w=wv'-uw_1, \quad \delta_3w=uu'-vv',\\ &\delta_1^2V=v^2\bar{w}_1-2vw\bar{u}'+w^2\bar{v}_1 \qquad +\bar{w}(vu'-wv_1)-\bar{v}(vw_1-wu'),\\ &\delta_1\delta^2V=uw\bar{u}'+vw\bar{v}'-w^2\bar{w}'-uv\bar{w}_1+\bar{u}(vw_1-wu')-\bar{w}(vv'-ww'),\\ &\delta_2\delta_1V=uw\bar{u}'+vw\bar{v}'-w^2\bar{w}'-uv\bar{w}_1+\bar{w}(ww'-uu')-\bar{v}(wv'-uw_1),\\ &\delta_2^2V=w^2\bar{u}_1-2wu\bar{v}'+u^2\bar{w}_1 \qquad +\bar{u}(wv'-uw_1)-\bar{w}(wu_1-uv'),\end{aligned}\right\} \quad . \quad . \quad (30)$$

each of which consists of two parts; viz. the first involves the second differential coefficients of V, the second the first differential coefficients of V. This being so, the first part of the coefficient of c^2, viz. $(\delta_1 V\delta_2-\delta_2 V\delta_1)^2 V$, will be

$$
\begin{aligned}
&=(w\bar u-u\bar w)^2(v^2\bar w_1-2vw\bar u'+w^2\bar v_1)\\
&\quad-2(w\bar u-u\bar w)(v\bar w-w\bar v)(wu\bar u'+vw\bar v'-w^2\bar w'-uv\bar w_1)\\
&\quad+(v\bar w-w\bar v)^2(w^2\bar u_1-2wu\bar v'+u^2\bar w_1)\\
&=w^2\{(v\bar w-w\bar v)^2\bar u_1+(w\bar u-u\bar w)^2\bar v_1+(u\bar v-v\bar u)^2 w_1\\
&\qquad+2(w\bar u-u\bar w)(u\bar v-v\bar u)u'\\
&\qquad+2(u\bar v-v\bar u)(v\bar w-w\bar v)v'\\
&\qquad+2(v\bar w-w\bar v)(w\bar u-u\bar w)w'\}
\end{aligned}
$$

$$
=w^2\begin{vmatrix}
\bar u_1, & \bar w', & \bar v', & \bar u, & u,\\
\bar w', & \bar v_1, & \bar u', & \bar v, & v,\\
\bar v', & \bar u', & \bar w_1, & \bar w, & w,\\
\bar u, & \bar v, & \bar w, & . & .,\\
u, & v, & w, & . & .,
\end{vmatrix}
=\frac{w^2t^2}{(m-1)^2}\begin{vmatrix}
\bar u_1, & \bar w', & \bar v', & \bar l', & u,\\
\bar w', & \bar v_1, & \bar u', & \bar m', & v,\\
\bar v', & \bar u', & \bar w_1, & \bar u', & w,\\
\bar l', & \bar m', & \bar u', & \bar k_1, & k,\\
u, & v, & w, & k, & .
\end{vmatrix}
=\frac{w^2t^2}{(m-1)^2}\bar\Omega \text{ suppose;}
$$

and the second part of the same expression

$$
\begin{aligned}
&=(w\bar u-u\bar w)^2\{-w\bar w v_1+(v\bar w+w\bar v)u'-v\bar w w_1\}\\
&\quad-(w\bar u-w\bar w)(v\bar w-w\bar v)\{-(w\bar u+u\bar w)u'-(v\bar w+w\bar v)v'+2w\bar w w'+(u\bar v+v\bar u)\bar w_1\}\\
&\quad+(v\bar w-w\bar v)^2\{-u\bar u w_1+(w\bar u+u\bar w)v'-w\bar w u_1\}\\
&\qquad=-w\bar w\{(v\bar w-w\bar v)^2 u_1+(w\bar u-u\bar w)^2 v_1+(u\bar v-v\bar u)^2 w_1\\
&\qquad\quad+2(w\bar u-u\bar w)(u\bar v-v\bar u)u'\\
&\qquad\quad+2(u\bar v-v\bar u)(v\bar w-w\bar v)v'\\
&\qquad\quad+2(v\bar w-w\bar v)(w\bar u-uw)w'\}
\end{aligned}
$$

$$
=-\frac{w\bar w t^2}{(u-1)^2}\begin{vmatrix}
u_1, & w', & v', & \bar l', & u\\
w', & v_1, & u', & m', & \bar v\\
v', & u', & w_1, & u', & \bar w\\
l', & m', & u', & k_1, & k\\
\bar u, & \bar v, & \bar w, & \bar k, & .
\end{vmatrix}
=-\frac{w\bar w t^2}{(u-1)^2}\Omega \text{ suppose.}
$$

Hence the whole expression (29)

$$
=(au+bv+cw)t^2\left\{(au+bv+cw)\frac{\bar\Omega}{(m-1)^2}-(a\bar u+b\bar v+c\bar w)\frac{\Omega}{(u-1)^2}\right\}.
$$

Hence the condition for a three-pointic contact in some plane about a given axis will be

$$
(au+bv+cw)\frac{\bar\Omega}{(m-1)^2}-(a\bar u+b\bar v+c\bar w)\frac{\Omega}{(u-1)^2}=0. \quad . \quad . \quad . \quad . \quad . \quad (31)
$$

This may be regarded as a condition to be fulfilled either by a, b, c, the direction cosines of the axis, or by x, y, z, t, the coordinates of the point. Taking the first view, $au+bv+cw=0$ is the condition that the axis shall lie in the tangent plane of U, and $a\bar{u}+b\bar{v}+c\bar{w}=0$ the condition that it shall lie in the tangent plane of V; hence (31) expresses the condition that the axis shall lie in the intersection of these planes.

On the other hand, regarding a, b, c as given, the equation (31) will represent a surface whose intersections with U and V will determine the points of three-pointic contact about a given axis. The degree of this surface is $3(m+n-3)$; and the number of points will therefore be $3mn(m+n-3)$.

Lastly, the equation (31) becomes independent of a, b, c if

$$\bar{\Omega}=0, \quad \Omega=0, \qquad (32)$$

which will consequently express the conditions that a three-pointic contact may subsist in some plane about any axis. The degrees of these equations are $2(n-1)+3(m-2)$, and $2(m-1)+3(n-2)$ respectively. Points for which such contact will subsist for any axis do not in general exist when U and V do not touch; but the condition for their existence will be found by eliminating x, y, z, t from the equations $U=0$, $V=0$, $\Omega=0$, $\bar{\Omega}=0$.

§ 3. *Modes of Contact.*

Hitherto we have considered only the contact of the curves of section of the surfaces U, V made by definite planes passing through an axis. If, however, in the equations (18), which express the conditions for the contact of these curves, we equate to zero the coefficients of the various powers of the quantity $\varpi':\varpi$, which determines the azimuth, we shall obtain a new series of conditions. And the fulfilment of these conditions will ensure the subsistence of contact, of the degree under consideration, independently of the azimuth of the cutting plane; or, in other words, for all plane sections round the point P whose planes of section pass through the axis, such contact may be called *circumaxal*; and, in particular, contact which holds good in this manner for a single axis might be termed *uniaxal contact*; that which holds good similarly for two axes might be termed *biaxal contact*; and so on for a greater number of axes. But before entering into this question, it will be as well to establish a theorem relating to the number of sections necessary to ensure uniaxal contact.

Returning to equations (18); the second, viz $(\varpi'\square-\varpi\square')V=0$, expresses the condition for two-pointic contact. Suppose that this holds good for more than one value of $\varpi':\varpi$, say, $\varpi'_1:\varpi_1$ and $\varpi'_2:\varpi_2$. Then, writing down the equation for each of these values, we may eliminate the coefficients and obtain the resultant,

$$\varpi_2\varpi'_2-\varpi_2\varpi'_1=0. \qquad (33)$$

But as $\varpi'_1:\varpi_1$ and $\varpi'_2:\varpi_2$ are by hypothesis different, the above equation cannot be satisfied, and consequently the coefficients of ϖ' and ϖ in the equation under consideration must separately vanish. But the evanescence of these coefficients expresses the conditions for universal two-pointic contact. Hence if a two-pointic contact subsists for two positions of the cutting plane about an axis, it will subsist for all positions about that axis. It will be shown in the sequel, as is well known from other considerations, that under these circumstances the contact will hold good for all axes through the point P. A similar result follows in the case of three-pointic contact. If the third equation of (18) holds good for three values of $\varpi' : \varpi_1$, say $\varpi'_1 : \varpi_1$; $\varpi'_2 : \varpi_2$; $\varpi'_3 : \varpi_3$, then writing down the equation for the three values successively, we shall be able to eliminate the three coefficients of the powers of $\varpi' : \varpi$ and obtain the resultant,

$$\begin{vmatrix} \varpi_1'^2, & \varpi_1'\varpi_1, & \varpi_1^2, \\ \varpi_2'^2, & \varpi_2'\varpi_2, & \varpi_2^2, \\ \varpi_3'^2, & \varpi_3'\varpi_3, & \varpi_3^2 \end{vmatrix}$$

$$= -(\varpi_2\varpi'_3 - \varpi'_2\varpi_3)(\varpi_3\varpi'_1 - \varpi_1\varpi'_3)(\varpi_1\varpi'_2 - \varpi_2\varpi'_1) = 0, \quad \ldots \ldots \quad (34)$$

which cannot be satisfied, since by hypothesis the three values of $\varpi' : \varpi$ are all different. Hence the coefficients of the equation in question must separately vanish. In other words, if a three-pointic contact subsist for three positions of the cutting plane about an axis, it will subsist for all positions about that axis.

The same law may obviously be extended to contacts of higher degrees.

The axis may be drawn, as before stated, in any direction through the point P; it may therefore be made to coincide with a tangent to the curve of intersection of U and V at the point. But in that case it is obvious that two-pointic contact would subsist for two positions (in fact for all positions) of the cutting plane without involving the conditions for the ordinary contact of the two surfaces (viz. $\delta_1 V=0$, $\delta_2 V=0$, $\delta_3 V=0$) as a consequence. It is perhaps desirable to show that the formulæ here employed take cognizance of this circumstance, as well as of the corresponding circumstances in the cases of contact of higher degrees.

Suppose, then, that two-pointic contact subsists for two positions of the cutting plane about the axis, say for the two planes (A, B, C, D), (A_1, B_1, C_1, D_1); then, adopting the last form of the group (8), we have the two equations

$$\left.\begin{aligned} A\delta_1 V + B\delta_2 V + C\delta_3 V = 0, \\ A_1\delta_1 V + B_1\delta_2 V + C_1\delta_3 V = 0. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (35)$$

Adding to these the two identical equations,

$$u\delta_1 U + v\delta_2 V + w\delta_3 V = 0,$$
$$\bar{u}\delta_1 U + \bar{v}\delta_2 V + \bar{w}\delta_3 V = 0,$$

and eliminating $\delta_1 V$, $\delta_2 V$, $\delta_3 V$, we obtain the resultants

$$\begin{Vmatrix} u, & \bar{u}, & A, & A_1, \\ v, & \bar{v}, & B, & B_1, \\ w, & \bar{w}, & C, & C_1, \end{Vmatrix} = 0. \quad \ldots \ldots \ldots \quad (36)$$

And if we regard these equations as determining a particular direction for the axis, they express the condition that it must coincide with the tangent line to the curve of intersection of U and V at the point P; so that in this particular case the equations (35) do not involve $\delta_1 V=0, \ldots$ as a consequence.

Again, in the case of three-pointic contact, we may take the following form, viz.

$$A^2\delta_1^2 V+B^2\delta_2^2 V+\ldots+2BC\delta_2\delta_3 V+\ldots=0. \qquad (37)$$

Then, since the operation $u\delta_1+v\delta_2+w\delta_3$ vanishes identically, we obtain, by operating with it upon u, v, w respectively, and then eliminating u, v, w, the following resultant:—

$$\begin{vmatrix} \delta_1^2 V, & \delta_2\delta_1 V, & \delta_3\delta_1 V, \\ \delta_1\delta_2 V, & \delta_2^2 V, & \delta_3\delta_2 V, \\ \delta_1\delta_3 V, & \delta_2\delta_3 V, & \delta_3^2 V \end{vmatrix}=0. \qquad (38)$$

But this is the condition that (37) may be resolved into linear factors. Supposing it so resolved into the product $(AP+\ldots)(AP_1+\ldots)$, then one of these factors must vanish in virtue of (37). If, then, the contact subsists for three positions of the cutting plane, we may write

$$AP+\ldots=0, \quad A_1P+\ldots=0, \quad A_2P+\ldots=0; \qquad (39)$$

to which we may add, in virtue of the identical equations,

$$u^2\delta_1^2 V+v^2\delta_2^2 V+\ldots+2vw\delta_2\delta_3 V+\ldots=(uP+\ldots)(uP_1+\ldots)=0,$$
$$\bar{u}^2\delta_1^2 V+\bar{v}^2\delta_2^2 V+\ldots+2\bar{v}\bar{w}\delta_2\delta_3 V+\ldots=(\bar{u}P+\ldots)(\bar{u}P_1+\ldots)=0,$$

the following,

$$uP+\ldots=0, \quad \bar{u}P+\ldots=0; \qquad (40)$$

whence, eliminating P, .., we obtain

$$\left\|\begin{matrix} u, & \bar{u}, & A, & A_1, & A_2, \\ v, & \bar{v}, & B, & B_1, & B_2, \\ w, & \bar{w}, & C, & C_1, & C_2 \end{matrix}\right\|=0, \qquad (41)$$

showing that if the planes all intersect in a tangent to the curve of intersection, the conditions $\delta_1^2 V=0$, $\delta_2^2 V=0, \ldots$ are not of necessity fulfilled.

It is perhaps unnecessary to pursue this part of the subject further.

Returning from this digression to the equations (18), it may be observed that if there be two-pointic circumaxal contact about the point P, *i. e.* when $\Box V=0$, $\Box' V=0$, the equation $(\varpi'\Box-\varpi\Box')^2 V=0$ will be satisfied by two values of $\varpi':\varpi$; in other words, the curve of intersection of U and V will have a double point at P, and along each of the branches the contact will be three-pointic. Similarly, if there be three-pointic circumaxal contact about the point P, *i. e.* when in addition to the former ($\Box V=0$, $\Box' V=0$), we have $\Box^2 V=0$, $(\Box'\Box+\Box\Box')V=0$, $\Box'^2 V=0$, then the equation $(\varpi'\Box-\varpi\Box')^3 V=0$ will be satisfied by three values of $\varpi':\varpi$; that is, the curve of intersection will have a triple point at P, and along each of the branches the contact will be four-pointic. This may be extended generally; but there will be occasion to return to the question hereafter*.

It having been proved that if p-pointic contact subsist for p plane sections about an axis it will subsist for all plane sections about that axis, the question naturally suggests itself whether there be not a corresponding theory as to the number of axes about which there must be circumaxal contact in order that it may subsist for all axes. In uniaxal contact it is supposed that from the point P at which the surfaces meet a series of curves are drawn (on both surfaces) lying in planes passing through the axis, and that contact of the degree under consideration subsists between every curve on U and the corresponding curve on V. If the circumaxal contact be multiaxal, we are supposed to take other axes through P, and draw other series of curves in planes passing through these axes respectively; and the question is, whether it be necessary that the contact shall subsist for a definite number of these series of curves, in order that it may subsist for all such series. In the latter case we shall call the contact *superficial*; commencing with two-pointic contact, and taking the form $A\delta_1+B\delta_2+C\delta_3$ for $\square$, we obtain, on equating to zero the coefficients of ϖ', ϖ respectively,

$$\left.\begin{aligned}\alpha\delta_1V+\beta\delta_2V+\gamma\delta_3V=0,\\ \alpha'\delta_1V+\beta'\delta_2V+\gamma'\delta_3V=0;\end{aligned}\right\} \quad \ldots \ldots \quad (42)$$

and applying to these the transformation (14), we deduce the following forms,

$$(b\delta_3-c\delta_2)V=0,\quad (c\delta_1-a\delta_2)V=0,\quad (a\delta_2-b\delta_1)V=0,$$

or

$$\delta_1V:\delta_2V:\delta_3V=a:b:c. \quad \ldots \ldots \quad (43)$$

If this contact holds good for a second axis (say a_1, b_1, c_1), we shall have also

$$\delta_1V:\delta_2V:\delta_3V=a_1:b_1:c_1. \quad \ldots \ldots \quad (44)$$

But since the two axes by hypothesis do not coincide, (43) and (44) cannot both be satisfied except on the conditions

$$\delta_1V=0,\quad \delta_2V=0,\quad \delta_3V=0. \quad \ldots \ldots \quad (45)$$

These conditions are in reality only two in number, in consequence of the identical relation $u\delta_1+v\delta_2+w\delta_3=0$. This shows that if two-pointic contact be biaxal it will be superficial. But inasmuch as the directions of these axes are arbitrary, we may take for one of them the tangent to the curve of intersection of U and V through P; hence, setting this axis aside, and reckoning only arbitrary axes, we may state that if two-pointic contact be uniaxal it will be superficial. This, of course, is merely the ordinary property of common contact.

At the risk of being tedious on so simple a question, I venture to point out that a result substantially the same may be deduced by a geometrical consideration from the equation $(A\delta_1+B\delta_2+C\delta_3)V=0$, without the intervention of the suppositions (45). Take

* Since writing the above I have found that a similar theorem was enunciated by M. MOUTARD. PONCELET, 'Application d'Analyse à la Géométrie,' 1864, tom. ii. p. 363.

two axes passing through P (say PQ, PQ_1), and a pair of planes passing through each (say PQQ_1, PQQ_2, and PQ_1Q, PQ_1Q_2); then, if two-pointic contact subsist for each pair of planes, the contact will be biaxal, as was shown at the commencement of the present section. We shall now have three planes in all, PQ_1Q_2, PQ_2Q, PQQ_1 (say the planes A, B, C; A_1, B_1, C_1; A_2, B_2, C_2), forming a solid angle; and in virtue of the equation with which we started, we shall have

$$\left.\begin{aligned} A\,\delta_1V+B\,\delta_2V+C\,\delta_3V&=0,\\ A_1\delta_1V+B_1\delta_2V+C_1\delta_3V&=0,\\ A_2\delta_1V+B_2\delta_2V+C_2\delta_3V&=0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

But as these planes by hypothesis do not pass through one and the same straight line, the determinant of these equations cannot vanish. Hence the system (46) can hold good only on the conditions $\delta_1V=0$, $\delta_2V=0$, $\delta_3V=0$. But we may take, as before, the tangent to the curve of intersection at P as one of the axes PQ, PQ_1. Hence we come to the same conclusion as before.

Passing to the case of three-pointic contact (and supposing that two-pointic superficial contact subsists at the point P), and equating to zero the coefficients of the powers of $\varpi':\varpi$ in the equation $(\varpi'\square-\varpi\square')^3V=0$, and adopting the same form as before, we shall obtain

$$\left.\begin{aligned} &\alpha^2\,\delta_1^2V+\beta^2\,\delta_2^2V+\ldots+2\beta\gamma\delta_2\delta_3V+\ldots=0,\\ &\alpha\alpha'\delta_1^2V+\beta\beta'\delta_2^2V+\ldots+(\beta\gamma'+\beta'\gamma)\delta_2\delta_3V+\ldots=0,\\ &\alpha'^2\delta_1^2V+\beta'^2\,\delta_2^2V+\ldots+2\beta'\gamma'\delta_2\delta_3V+\ldots=0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (47)$$

which, by means of the transformation (16), may be reduced to the following forms:—

$$\left.\begin{aligned} (c\delta_2-b\delta_3)^2V=0,\quad (a\delta_3-c\delta_1)^2V=0,\quad (b\delta_1-a\delta_2)^2V=0,\\ (a\delta_3-c\delta_1)(b\delta_1-a\delta_2)V=0,\\ (b\delta_1-a\delta_2)(c\delta_2-b\delta_3)V=0,\\ (c\delta_2-b\delta_3)(a\delta_3-c\delta_1)V=0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (48)$$

whereof three only are independent.

And if the contact be triaxal, we should have (taking the first of these forms)

$$\left.\begin{aligned} c^2\delta_2^2V-2b\,c\,\delta_2\delta_3V+b^2\delta_3^2V&=0,\\ c_1^2\delta_2^2V-2b_1c_1\delta_2\delta_3V+b_1^2\delta_3^2V&=0,\\ c_2^2\delta_2^2V-2b_2c_2\delta_2\delta_3V+b_2^2\delta_3^2V&=0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Eliminating the coefficients, we obtain, by the usual method,

$$(b_1c_2-b_2c_1)(b_2c-bc_2)(bc_1-b_1c)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

But as by hypothesis the three axes are all distinct, this equation cannot be satisfied; and therefore (49) can coexist only on the conditions $\delta_2^2V=0$, $\delta_2\delta_3V=0$, $\delta_3^2V=0$. Hence if the contact be triaxal it will be superficial. But we may take for two of the axes of triaxal contact the tangents to the two branches of the curve of intersection through P; and for every position of the cutting plane about each of these axes the contact will be three-pointic, viz. two consecutive points of the branch to which the axis is a tangent and one point of the other branch will lie in the plane; whence it follows that, reckoning only arbitrary axes as before, if three-pointic contact be uniaxal it will be superficial. And this method has application to all degrees of contact.

The equations (48) would apparently determine two axes about which three-pointic contact would be circumaxal; but that this is not the case will appear from the actual solution of one of them. In fact the solution of the third equation depends upon the quantity $(\delta_1\delta_2 V)^2-\delta_1^2V\delta_2^2V$, in order to develop which we have the following values:—

$$\delta_1^2 V=-\left|\begin{array}{ccc} \bar{v}_1, & \bar{u}', & v, \\ u', & \bar{w}_1, & w, \\ v, & w, & . \end{array}\right| +\theta \left|\begin{array}{ccc} v_1, & u', & v, \\ u', & w_1, & w, \\ v, & w, & . \end{array}\right|$$

$$-\delta_1\delta_2 V=-\left|\begin{array}{ccc} \bar{w}', & \bar{v}', & u, \\ \bar{u}', & \bar{w}_1, & w, \\ v, & w, & . \end{array}\right| +\theta \left|\begin{array}{ccc} w', & v', & u, \\ u', & w_1, & w, \\ v, & w, & . \end{array}\right|$$

$$\delta_2^2 V=-\left|\begin{array}{ccc} \bar{u}_1, & \bar{v}', & u, \\ \bar{v}', & \bar{w}_1, & w, \\ u, & w, & . \end{array}\right| +\theta \left|\begin{array}{ccc} u_1, & v', & u, \\ v', & w_1, & w, \\ u, & w, & . \end{array}\right|$$

Hence, by the method of compound determinants, in the expression $(\delta_1\delta_2V)^2-\delta_1^2V\delta_2^2V$, the term independent of θ

$$=w^2\left|\begin{array}{cccc} \bar{u}_1, & \bar{w}', & \bar{v}', & u, \\ \bar{w}', & \bar{v}_1, & \bar{u}', & v, \\ \bar{v}', & \bar{u}', & \bar{w}_1, & w, \\ u, & v, & w, & . \end{array}\right| =-w^2(\bar{v}_1\bar{w}_1-\bar{u}'^2, \ldots)(u, v, w)^2,$$

the coefficient of θ^2

$$=w^2\left|\begin{array}{cccc} u_1, & w', & v', & u, \\ w', & v_1, & u', & v, \\ v', & u', & w_1, & w, \\ u, & v, & w, & . \end{array}\right| =-w^2(v_1w_1-u'^2, \ldots)(u, v, w)^2;$$

while the coefficient of θ will be found to be

$$=w^2(v_1\bar{w}_1+w_1\bar{v}_1-2u'\bar{u}', \ldots)(u, v, w)^2;$$

so that the whole expression sought

$$\begin{aligned} = \quad & -w^2\{(\bar{v}_1\bar{w}_1-\bar{u}'^2, \ldots)(u, v, w)^2 \\ & -(v_1\bar{w}_1+w_1\bar{v}_1-2u'\bar{u}', \ldots)(u, v, w)(\bar{u}, \bar{v}, \bar{w}) \\ & +(v_1w_1-u'^2, \ldots)(\bar{u}, \bar{v}, \bar{w})^2\} \\ = & -w^2\Phi \text{ suppose.} \end{aligned}$$

This being the case, the solutions of the equations

$$(b\delta_3-c\delta_2)^2\text{V}=0, \quad (c\delta_1-a\delta_3)^2\text{V}=0, \quad (a\delta_2-b\delta_1)^2\text{V}=0$$

may be written in the following forms:—

$$\left.\begin{aligned} b\delta_3^2\text{V}-c\delta_2^2\text{V} &= \pm u\,c\,\sqrt{-\Phi}, \quad = \mp u\,b\,\sqrt{-\Phi}, \\ c\delta_1^2\text{V}-a\delta_3^2\text{V} &= \pm v\,a\,\sqrt{-\Phi}, \quad = \mp v\,c\,\sqrt{-\Phi}, \\ a\delta_2^2\text{V}-b\delta_1^2\text{V} &= \pm wb\,\sqrt{-\Phi}, \quad = \mp wa\,\sqrt{-\Phi}, \end{aligned}\right\} \quad \ldots\ldots \quad (51)$$

which involve $\Phi=0$. Φ is therefore a surface which cuts U in a curve, at each point of which there is an axis,

$$a:b:c=\delta_1^2\text{V}:\delta_2^2\text{V}:\delta_3^2\text{V},$$

about which there is three-pointic contact.

It may be shown also, by the following geometrical construction, that if three-pointic contact be triaxal it will be superficial. If we take three axes, PQ, PQ_1, PQ_2, and draw through each three planes; then if three-pointic contact subsist for each triplet of planes, the contact will be circumaxal for each axis, and therefore triaxal. If we take a fourth axis, PQ_3, the following planes will pass three and three through each of the axes, and will serve for the planes required, viz. the planes

$$\text{PQ}_1\text{Q}_2, \quad \text{PQ}_2\text{Q}, \quad \text{PQQ}_1, \quad \text{PQQ}_3, \quad \text{PQ}_1\text{Q}_3, \quad \text{PQ}_2\text{Q}_3,$$

say the planes (A, B, C), .. (A_5, B_5, C_5). Taking the forms $\text{A}\delta_1+\text{B}\delta_2+\text{C}\delta_3$ for $\square$, the conditions for three-pointic contact along each of these planes will be

$$\left.\begin{aligned} (\text{A}\delta\ +\text{B}\ \delta_2+\text{C}\ \delta_3)^2\text{V}=0, \quad (\text{A}_3\delta_1+\text{B}_3\delta_2+\text{C}_3\delta_3)^2\text{V}=0, \\ (\text{A}_1\delta_1+\text{B}_1\delta_2+\text{C}_1\delta_3)^2\text{V}=0, \quad (\text{A}_4\delta_1+\text{B}_4\delta_2+\text{C}_4\delta_3)^2\text{V}=0, \\ (\text{A}_2\delta_1+\text{B}_2\delta_2+\text{C}_2\delta_3)^2\text{V}=0, \quad (\text{A}_5\delta_1+\text{B}_5\delta_2+\text{C}_5\delta_3)^2\text{V}=0. \end{aligned}\right\} \quad \ldots\ldots \quad (52)$$

Eliminating $\delta_1^2\text{V}$, $\delta_2^2\text{V}$, .. $\delta_2\delta_3\text{V}$, .., we obtain the resultant,

$$\begin{vmatrix} \text{A}^2, & \text{B}^2, \ldots \text{B C}, \ldots \\ \text{A}_1^2, & \text{B}_1^2, \ldots \text{B}_1\text{C}_1, \ldots \\ \vdots & \\ \text{A}_5^2, & \text{B}_5^2, \ldots \text{B}_5\text{C}_5, \ldots \end{vmatrix} = 0, \quad \ldots\ldots\ldots \quad (53)$$

which is the condition that the six planes should all touch a cone of the second degree. But the planes in question pass three and three through four lines; and as it is impossible through any one line to draw more than two planes touching a cone of the second degree, the equations above written (52) cannot be satisfied except on the conditions $\delta_1^2V=0$, $\delta_2^2V=0$, .. $\delta_2\delta_3V=0$, .., which are in fact the conditions for superficial contact.

There is another more general way in which the subject may be regarded. In fact if α, β, γ (or, if we prefer so to state it, if A, B, C) no longer have the significations originally given to them, but represent the differential coefficients of an auxiliary surface W; say, if

$$\partial_x W=u, \quad \partial_y W=v, \quad \partial_z W=w, \quad \partial_t W=k, \quad \dots \quad (54)$$

then the equations

$$V=0, \quad (u\delta_1+v\delta_2+w\delta_3)V=0, \quad (u\delta_1+v\delta_2+w\delta_3)^2V=0, \quad \dots \quad (55)$$

will no longer express the conditions for two-, three-, .. pointic contact of the curves of section made by the plane (α, β, γ) or the plane (A, B, C), but the contact of the curves of section made by the surface W. And as the surface W is perfectly arbitrary, the formulæ will apply to any curve drawn at pleasure from the point P on the surface U. It is to be borne in mind that in expanding the expression for three-pointic contact we shall obtain

$$(u\delta_1+\dots)^2=(u\delta_1+\dots)u\delta_1V+(u\delta_1+\dots)v\delta_2V+\dots$$
$$+(u^2\delta_1^2+V^2\delta_2^2+\dots 2uv\delta_1\delta_2+\dots)V;$$

but in the only case which possesses much interest, viz. when the two-pointic contact at the point P is superficial, we have $\delta_1V=0$, $\delta_2V=0$, $\delta_3V=0$; and consequently

$$(u\delta_1+\dots)^2V=(u^2\delta_1^2+v^2\delta_2^2+\dots 2uv\delta_1\delta_2+\dots)V, \quad \dots \quad (56)$$

which is of the same form as the expression derived in the case of plane sections. And as the operators δ_1, δ_2, .. are unchanged, and are subject to the same identical relations as before, the conditions of contact now considered will be susceptible of the same transformations (the transformations (13–16) excepted) as those considered before. From these, therefore, we may draw the following conclusion:—

Consider two surfaces, U, V, having superficial two-, three-, .. pointic contact at a point P; from P draw any number of curves arbitrarily on U; two, three, .. consecutive points of these curves will, in consequence of the superficial contact, lie also on V. This being so, if for any three, four, .. of the curves an additional consecutive point lies on V, then the same will be the case for all the curves, and there will be superficial three-, four-, .. pointic contact between U and V at the point P.

This may be also stated in the following form:—

If two surfaces, U, V, have two-, three-, .. pointic superficial contact at a point P, and if through P we draw any number of surfaces arbitrarily, the curves of section on U and V which correspond to one another will, in consequence of the superficial contact, have two-, three-, .. pointic contact. This being the case, if any three-, four-, .. corresponding curves have three-, four-, .. pointic contact, then all will have three-, four-, .. pointic contact; and there will be three-, four-, .. pointic superficial contact between U and V at the point P.

This theorem for the case of three-pointic contact was given by DUPIN, 'Développements de Géométrie,' p. 12.

§ 4. *On Surfaces having Superficial Contact with given Surfaces.*

It is well known that at any point P of a surface U we may in general determine a plane V touching, or, in terms of this memoir, having two-pointic superficial contact with U. This suggests the question whether surfaces V of other degrees may not be determined having superficial contact of higher degrees with U at a point P.

The number of conditions for a 1, 2, .. p-pointic superficial contact has been shown above to be

$$1,\ 3,\ 6,\ ..\frac{p(p+1)}{1.2}.$$

Now the number of independent constants in the equation of a surface V of the degree 1, 2, .. m, is

$$3,\ 9,\ 19,\ ..\frac{(m+1)(m+2)(m+3)}{1.2.3}-1;$$

so that, employing the equations which express the conditions of contact for determining the constants of V, we shall meet with the following cases. First, if the number of conditions be equal to the number of constants, there will be a determinate surface V having a superficial contact of the degree under consideration (say p) with U at the point P. Secondly, if the number of conditions exceed that of the constants by unity, the constants may be eliminated, and the result will be an equation between the coordinates; in other words, an equation to a surface which will cut U in a curve at every point of which a surface V may be drawn having p-pointic superficial contact with U. Thirdly, if the number of conditions exceed that of the constants by 2, we may eliminate the constants in two ways, and obtain two resulting equations, which will represent two surfaces mutually cutting U in a finite number of points, at each of which a surface V may be drawn having p-pointic superficial contact with U. Lastly, if the number of conditions exceed that of the constants by more than two, we shall obtain a number of resulting equations equal to that excess. From these, together with the equations $U=0$, $V=0$, the variables may be eliminated; so that the number of resultants less 2 will represent the number of conditions which must subsist among the constants of U, in order that it may be possible to draw such a surface V. This being the case, there will be a determinate surface V of the degree m having p-pointic superficial contact with U, (1) at any point on U, or (2) along a certain curve on U, or (3) only at a finite number of points on U, according as the expression

$$\frac{p(p+1)}{1.2}-\frac{(m+1)(m+2)(m+3)}{1.2.3}+1=0,\ 1,\ 2;$$

or, clearing denominators, according as

$$3p(p+1)-m^3-6m^2-11m=0,\ 6,\ 12. \quad \text{. . . . . . .} \quad (57)$$

Now it is obvious from the signs of the terms on the left-hand side of this equation that the result cannot be positive if p be less than $2m$; beginning therefore with $p=2m$, we obtain

$$-m^3+6m^2-5m=0,\ 6,\ 12;$$

or, resolving into factors, we have the three cases

$$m(m-1)(m-5)=0,$$
$$(m-2)(m^2-4m+3)=0,$$
$$(m-3)(m-4)(m+1)=0.$$

Next, let $p=2m+1$; then, substituting this value in the equation (57), we obtain

$$-m^3+6m^2+7m+6=0,\ 6,\ 12;$$

or resolving into factors so far as possible, we have the three cases

$$m^3-6m^2-7m-6=0,$$
$$(m-7)(m+1)=0,$$
$$m^3-6m^2-7m+6=0,$$

the first and last of which give no solutions in positive whole numbers.

This appears to exhaust all the solutions of (57) in positive whole numbers. Recapitulating the foregoing results, we may form the following Table:—

Degree of contact.	Number of conditions.	Degree of V.	Number of constants.	Difference conds.–consts.	Superficial contact possible.
2	3	1	3	0	At every point on U.
4	10	2	9	1	Along a curve on U.
6	21	3	19	2	At a finite number of points on U.
8	36	4	34	2	
10	55	5	55	0	At every point on U.
15	120	7	119	1	Along a curve on U.

Such is the general theory. But it is probable that it undergoes modifications in each particular case; it certainly does so in the only case fully examined here, viz. that of a quadric having four-pointic contact with U.

In fact, inasmuch as the equations, whereby the constants are ultimately determined, are linear the solution is in every case unique. But four-pointic contact will subsist if we consider the quadric to consist of the tangent plane taken twice; and as the solution is unique, no other quadric can in general be drawn having four-pointic contact. Further, since a tangent plane can in general be drawn at every point on U, the quadric of four-pointic contact (viz. the tangent plane taken twice) exists generally; and the condition (viz. difference, conditions$-$constants$=1$) restricting it to a curve on U must be satisfied identically. Similar remarks apply to contact of the degrees 6, 8, 10; but not apparently to that of the degree 15.

It will be worth while to examine the case of four-pointic contact more in detail; and for this purpose the special transformation of the operation $\square^2$V, employed in the memoir before quoted, appears to be best adapted. The following method of effecting that transformation is perhaps more expeditious and direct than the process used in the memoir itself.

Taking for $\square$ the form $A\delta_1+B\delta_2+C\delta_3$, we have

$$\square^2 V=\square\begin{vmatrix} A, & u, & \partial_x, \\ B, & v, & \partial_y, \\ C, & w, & \partial_z, \end{vmatrix}=\begin{vmatrix} \square A, & u, & \partial_x, \\ \square B, & v, & \partial_y, \\ \square C, & w, & \partial_z, \end{vmatrix}+\begin{vmatrix} A, & \square u, & \partial_x, \\ B, & \square v, & \partial_y, \\ C, & \square w, & \partial_z, \end{vmatrix}+\begin{vmatrix} A, & u, & \square\partial_x, \\ B, & v, & \square\partial_y, \\ C, & w, & \square\partial_z. \end{vmatrix}$$

But by the equations (11) of the memoir,

$$\varpi\square A=\begin{vmatrix} A, & u, & \alpha A-\alpha A, \\ B, & v, & \alpha B-\beta A, \\ C, & w, & \alpha C-\gamma A, \end{vmatrix}=-\begin{vmatrix} A, & u, & \alpha, \\ B, & v, & \beta, \\ C, & w, & \gamma, \end{vmatrix} A=-\varpi\begin{vmatrix} \alpha, & \alpha', & u, \\ \beta, & \beta', & v, \\ \gamma, & \gamma', & w, \end{vmatrix} A;$$

whence

$$\begin{vmatrix} \square A, & u, & \partial_x, \\ \square B, & v, & \partial_y, \\ \square C, & w, & \partial_z, \end{vmatrix} V=-\begin{vmatrix} \alpha, & \alpha', & u, \\ \beta, & \beta', & v, \\ \gamma, & \gamma', & w, \end{vmatrix}\begin{vmatrix} A, & u, & \partial_x, \\ B, & v, & \partial_y, \\ C, & w, & \partial_z, \end{vmatrix} V=-\begin{vmatrix} \alpha, & \alpha', & u, \\ \beta, & \beta', & v, \\ \gamma, & \gamma', & w, \end{vmatrix}\square V=0,$$

since by hypothesis $\square V=0$. Again,

$$\square u=\begin{vmatrix} A, & u, & u_1, \\ B, & v, & w', \\ C, & w, & v', \end{vmatrix},\quad \square v=\begin{vmatrix} A, & u, & w', \\ B, & v, & v_1, \\ C, & w, & u', \end{vmatrix},\quad \square w=\begin{vmatrix} A, & u, & v', \\ B, & v, & u', \\ C, & w, & w_1; \end{vmatrix}$$

whence, remembering that (on the supposition $\square V=0$) $\partial_x V=\theta u$, $\partial_y V=\theta v$, $\partial_z V=\theta w$, we derive the following:

$$\begin{vmatrix} A, & \square u, & \partial_x, \\ B, & \square v, & \partial_y, \\ C, & \square w, & \partial_z, \end{vmatrix} V=\theta\begin{vmatrix} A, & \square u, & u, \\ B, & \square v, & v, \\ C, & \square w, & w, \end{vmatrix}=-\theta\begin{vmatrix} u_1, & w', & v', & u, & A, \\ w', & v_1, & u', & v, & B, \\ v', & u', & w_1, & w, & C, \\ u, & v, & w, & . & . \\ A, & B, & C, & . & . \end{vmatrix}=-\frac{\theta t^2}{(n-1)^2}H,$$

where

$$\mathrm{H}=\begin{vmatrix} u_1, & w', & v', & l', & \mathrm{A}, \\ w', & v'_1, & u', & m', & \mathrm{B}, \\ v', & u', & w_1, & u', & \mathrm{C}, \\ l', & m', & u', & k_1, & \mathrm{D}, \\ \mathrm{A}, & \mathrm{B}, & \mathrm{C}, & \mathrm{D}, & . \end{vmatrix}.$$

Again, if in the following expressions ∂_x, ∂_y, ∂_z, ∂_t be understood to affect V alone, then

$$\begin{vmatrix} \mathrm{A}, & u, & \square\partial_x, \\ \mathrm{B}, & v, & \square\partial_y, \\ \mathrm{C}, & w, & \square\partial_z, \end{vmatrix}\mathrm{V}=\begin{vmatrix} u_1, & w', & v', & u, & \mathrm{A}, & \partial_x, \\ w', & v_1, & u', & v, & \mathrm{B}, & \partial_y, \\ v', & u', & w_1, & w, & \mathrm{C}, & \partial_z, \\ u, & v, & w, & . & . & . \\ \mathrm{A}, & \mathrm{B}, & \mathrm{C}, & . & . & . \\ \partial_x, & \partial_y, & \partial_z, & . & . & . \end{vmatrix}\mathrm{V}=\frac{t}{n-1}\begin{vmatrix} u_1, & w', & v', & l', & \mathrm{A}, & \partial_x, \\ w', & v_1, & u', & m', & \mathrm{B}, & \partial_y, \\ v', & u', & w_1, & u', & \mathrm{C}, & \partial_z, \\ u, & v, & w, & k, & . & . \\ \mathrm{A}, & \mathrm{B}, & \mathrm{C}, & \mathrm{D}, & . & . \\ \partial_x, & \partial_y, & \partial_z, & \partial_t-\frac{\mathrm{D}}{t}, & . & . \end{vmatrix}\mathrm{V},$$

where $\mathrm{D}=x\partial_x+y\partial_y+z\partial_z+t\partial_t$. Also, by similar processes, the above expression may be further transformed as follows:

$$=\frac{t^2}{(n-1)^2}\begin{vmatrix} u_1, & w', & v', & l', & \mathrm{A}, & \partial_x, \\ w', & v_1, & u', & m', & \mathrm{B}, & \partial_y, \\ v', & u', & w_1, & u', & \mathrm{C}, & \partial_z, \\ l', & m', & u', & k_1, & \mathrm{D}, & \partial_t-\frac{\mathrm{D}}{t}, \\ \mathrm{A}, & \mathrm{B}, & \mathrm{C}, & \mathrm{D}, & . & . \\ \partial_x, & \partial_y, & \partial_z, & \partial_t-\frac{\mathrm{D}}{t}, & . & . \end{vmatrix}\mathrm{V}=\frac{t^2}{(n-1)^2}\left(\Delta\mathrm{V}-2\,\frac{m-1}{n-1}\,\theta\mathrm{H}\right),$$

where

$$\Delta=\begin{vmatrix} u_1, & w', & v', & l', & \mathrm{A}, & \partial_x, \\ w', & v_1, & u', & m', & \mathrm{B}, & \partial_y, \\ v', & u', & w_1, & u', & \mathrm{C}, & \partial_z, \\ l', & m', & u', & k_1, & \mathrm{D}, & \partial_t, \\ \mathrm{A}, & \mathrm{B}, & \mathrm{C}, & \mathrm{D}, & . & . \\ \partial_x, & \partial_y, & \partial_z, & \partial_t, & . & . \end{vmatrix}$$

So that finally the expression $\square^2\mathrm{V}=0$, combined with $\mathrm{V}=0$, $\square\mathrm{V}=0$, is reduced to

$$\Delta\mathrm{V}-\left(1+2\frac{m-1}{n-1}\right)\theta\mathrm{H}=0,$$

where

$$\theta=\frac{\partial_x V}{u}=\frac{\partial_y V}{v}=\frac{\partial_z V}{w}=\frac{\partial_t V}{k},$$

which agree with the results obtained in the memoir.

In order to determine a quadric V which shall have superficial four-pointic contact with the surface U, let H_{11} be the value of H when $\alpha, \beta, \gamma, \delta$ are written for A, B, C, D respectively in the last line and the last column of H; let H_{12} be the value of H when $\alpha, \beta, \gamma, \delta$ are written for A, B, C, D respectively in the last line or the last column, and $\alpha', \beta', \gamma', \delta'$ in the last column or the last line of H; and let H_{22} be the value of H when $\alpha', \beta', \gamma', \delta'$ are written for A, B, C, D respectively in the last line and the last column of H. Further, let

$$\partial_x H=p, \quad \partial_y H=q, \quad \partial_z H=r, \quad \partial_t H=s, \qquad (58)$$

the differentiations being effected (as was shown in the memoir to be permissible) without reference to A, B, C, D. Lastly, if $p_{11}, \ldots, p_{12}, \ldots, p_{22}, \ldots$ be the values of $p, \ldots$ when H becomes H_{11}, H_{12}, H_{22}, respectively, let

$$X, Y, Z, T=\left\Vert\begin{array}{cccc} A, & B, & C, & D, \\ u, & v, & w, & k, \\ p, & q, & r, & s, \end{array}\right\Vert, \quad P, Q, R, S=-\left\Vert\begin{array}{ccc|cccc} A, & u, & p, & u_1, & w', & v', & l', \\ B, & v, & q, & w', & v_1, & u', & m', \\ C, & w, & r, & v', & u', & w_1, & u', \\ D, & k, & s, & l', & m', & u', & k_1. \end{array}\right\Vert \qquad (59)$$

That is to say, X, Y, Z, T are the determinants formed from the matrix opposite to them by omitting each of the columns in order; and P, Q, R, S are the negatives of the determinants formed from the matrix opposite to them by omitting each of the columns 4, 5, 6, 7 in order, and always retaining the columns 1, 2, 3.

This being premised, the conditions which the coefficients of the quadric

$$V=(a, b, c, d, f, g, h, l, m, n)(x, y, z, t)^2 \qquad (60)$$

must satisfy in order that four-pointic contact may subsist between the two curves of section of the surfaces U, V made by the plane $Ax+By+Cz+Dt=0$ will be, as proved in the memoir above quoted,

$$\left.\begin{array}{l} (uX-xP)a+(uY-yP)h+(uZ-zP)g+(uT-tP)l=0, \\ (vX-xQ)h+(vY-yQ)b+(vZ-zQ)f+(vT-tQ)m=0, \\ (wX-xR)g+(wY-yR)f+(wZ-zR)c+(wT-tR)n=0, \\ (kX-xS)l+(kY-yS)m+(kZ-zS)n+(kT-tS)d=0; \end{array}\right\} \qquad (61)$$

and the contact will be circumaxal if the foregoing equations are made independent of $\varpi' : \varpi$. If, therefore, we represent by $X_{111}, \ldots P_{111}, \ldots ; X_{112}, \ldots P_{112}, \ldots ; X_{122}, \ldots P_{122}, \ldots ; X_{222}, \ldots P_{222}, \ldots$ the coefficients of the powers of $\varpi' : \varpi$ in $X, \ldots P, \ldots$ respectively, we shall have four equations in the place of each one of the above group—apparently twelve equations in all. These, however, are equivalent to only nine, as may be thus shown. Taking the first, and equating to zero the coefficients of the several powers of $\varpi' : \varpi$, and eliminating a, h, g, l, we obtain the condition,

$$\begin{vmatrix} uX_{111}-xP_{111}, & uY_{111}-yP_{111}, & uZ_{111}-zP_{111}, & uT_{111}-tP_{111}, \\ uX_{112}-xP_{112}, & uY_{112}-yP_{112}, & uZ_{112}-zP_{112}, & uT_{112}-tP_{112}, \\ uX_{122}-xP_{122}, & uY_{122}-yP_{122}, & uZ_{122}-zP_{122}, & uT_{122}-tP_{122}, \\ uX_{222}-xP_{222}, & uY_{222}-yP_{222}, & uZ_{222}-zP_{222}, & uT_{222}-tP_{222}, \end{vmatrix}=0; \quad . \quad . \quad (62)$$

and it is not difficult to see that omitting the factor u^3, and employing the relations $u_1X+w'Y+v'Z+l'T=P$ &c., this expression may be reduced to the following form:—

$$\begin{vmatrix} u, & x, & y, & z, & t, \\ P_{111}, & X_{111}, & Y_{111}, & Z_{111}, & T_{111}, \\ P_{112}, & X_{112}, & Y_{112}, & Z_{112}, & T_{112}, \\ P_{122}, & X_{122}, & Y_{122}, & Z_{122}, & T_{122}, \\ P_{222}, & X_{222}, & Y_{222}, & Z_{222}, & T_{222}, \end{vmatrix} = u \begin{vmatrix} X_{111}, & Y_{111}, & Z_{111}, & T_{111}, \\ X_{112}, & Y_{112}, & Z_{112}, & T_{112}, \\ X_{122}, & Y_{122}, & Z_{122}, & T_{122}, \\ X_{222}, & Y_{222}, & Z_{222}, & T_{222}, \end{vmatrix} \quad . \quad (63)$$

which vanishes identically, since $uX_{111}+vY_{111}+wZ_{111}+kT_{111}=0$ &c.

The four equations derived from each of (61) are consequently reduced to three, and the whole number of equations connecting the ten coefficients of U to nine, the proper number for the determination of their nine ratios.

This being the case, we may take as the equations for determining the ratios a : h : g : l the following, viz.

$$(uX_{111}-xP_{111})a+(uY_{111}-yP_{111})h+(uZ_{111}-zP_{111})g+(uT_{111}-tP_{111})l=0,$$
$$(uX_{112}-xP_{112})a+(uY_{112}-yP_{112})h+(uZ_{112}-zP_{112})g+(uT_{112}-tP_{112})l=0,$$
$$(uX_{122}-xP_{122})a+(uY_{122}-yP_{122})h+(uZ_{122}-zP_{122})g+(uT_{122}-tP_{122})l=0;$$

and the quantity to which l is proportional will then be

$$\begin{vmatrix} uX_{111}-xP_{111}, & uY_{111}-yP_{111}, & uZ_{111}-zP_{111}, \\ uX_{112}-xP_{112}, & uY_{112}-yP_{112}, & uZ_{112}-zP_{112}, \\ uX_{122}-xP_{122}, & uY_{122}-yP_{122}, & uZ_{122}-zP_{122}, \end{vmatrix}$$

which, omitting the factor u^2, is equal to

$$\begin{vmatrix} u, & x, & y, & z, \\ P_{111}, & X_{111}, & Y_{111}, & Z_{111}, \\ P_{112}, & X_{112}, & Y_{112}, & Z_{112}, \\ P_{122}, & X_{122}, & Y_{122}, & Z_{122}, \end{vmatrix} = \begin{vmatrix} -(n-2)n+l't, & x, & y, & z, \\ l'T_{111}, & X_{111}, & Y_{111}, & Z_{111}, \\ l'T_{112}, & X_{112}, & Y_{112}, & Z_{112}, \\ l'T_{122}, & X_{122}, & Y_{122}, & Z_{122}, \end{vmatrix} = -(n-2)n \begin{vmatrix} X_{111}, & Y_{111}, & Z_{111}, \\ X_{112}, & Y_{112}, & Z_{112}, \\ X_{122}, & Y_{122}, & Z_{122}, \end{vmatrix}.$$

Calling the coefficient of $-(n-2)n$ in the last expression (X, Y, Z), and forming similar expressions in Y, Z, T, &c., we shall obtain the following values for the ratios sought, viz.—

$$\frac{\mathrm{a}}{(\mathrm{Y},\mathrm{Z},\mathrm{T})}=\frac{\mathrm{h}}{(\mathrm{Z},\mathrm{X},\mathrm{T})}=\frac{\mathrm{g}}{(\mathrm{X},\mathrm{Y},\mathrm{T})}=\frac{-l}{(\mathrm{X},\mathrm{Y},\mathrm{Z})} \quad \ldots\ldots\ldots \quad (64)$$

But

$$k(\mathrm{Y},\mathrm{Z},\mathrm{T})=-u(\mathrm{X},\mathrm{Y},\mathrm{Z}),$$
$$k(\mathrm{Z},\mathrm{X},\mathrm{T})=-v(\mathrm{X},\mathrm{Y},\mathrm{Z}),$$
$$k(\mathrm{X},\mathrm{Y},\mathrm{T})=-w(\mathrm{X},\mathrm{Y},\mathrm{Z}).$$

Hence, omitting the common factor (X, Y, Z), we have

$$\frac{\mathrm{a}}{u}=\frac{\mathrm{h}}{v}=\frac{\mathrm{g}}{w}=\frac{\mathrm{l}}{k}; \quad \ldots\ldots\ldots\ldots \quad (65)$$

and proceeding in a similar manner with the equations in h, b, f, m; g, f, c, n; l, m, n, d, it will be found that, when the quantity (X, Y, Z) does not vanish,

$$\mathrm{V}=(\mathrm{a},\mathrm{b},\ldots,\mathrm{f},\ldots\mathrm{l},\ldots)(x,y,z,t)^2=(ux+vy+wz+kt)^2; \quad \ldots \quad (66)$$

that is to say, that the only quadric which in general has a superficial four-pointic contact with U at any given point P is the tangent plane taken twice.

From the relations given above, it appears that if any one of the equations

$$(\mathrm{Y},\mathrm{Z},\mathrm{T})=0, \quad (\mathrm{Z},\mathrm{X},\mathrm{T})=0, \quad (\mathrm{X},\mathrm{Y},\mathrm{T})=0, \quad (\mathrm{X},\mathrm{Y},\mathrm{Z})=0 \quad \ldots \quad (67)$$

is satisfied, then all are satisfied; so that it will be sufficient to study any one of them.

Although I have not succeeded in reducing the expression (X, Y, Z) to any very simple form, it may still be worth while to show how it may be extricated from the condition of a compound determinant. With a view to this, we may write down the values of $\mathrm{X}_{111}, \ldots$ in full, viz.:—

$$\mathrm{X}_{111}=\begin{vmatrix}\beta, & \gamma, & \delta,\\ v, & w, & k,\\ q_{11}, & r_{11}, & s_{11},\end{vmatrix} \qquad \mathrm{Y}_{111}=\begin{vmatrix}\gamma, & \alpha, & \delta,\\ w, & u, & k,\\ r_{11}, & p_{11}, & s_{11},\end{vmatrix} \qquad \mathrm{Z}_{111}=\ldots, \ \mathrm{T}_{111}=\ldots$$

$$\mathrm{X}_{112}=2\begin{vmatrix}\beta, & \gamma, & \delta,\\ v, & w, & k,\\ q_{12}, & r_{12}, & s_{12},\end{vmatrix}+\begin{vmatrix}\beta', & \gamma', & \delta',\\ v, & w, & k,\\ q_{11}, & r_{11}, & s_{11},\end{vmatrix} \quad \mathrm{Y}_{112}=2\begin{vmatrix}\gamma, & \alpha, & \delta,\\ w, & u, & k,\\ r_{12}, & p_{12}, & s_{12},\end{vmatrix}+\begin{vmatrix}\gamma', & \alpha', & \delta',\\ w, & u, & k,\\ r_{11}, & p_{11}, & s_{11},\end{vmatrix} \quad \mathrm{Z}_{112}=\ldots, \ \mathrm{T}_{112}=\ldots$$

$$\mathrm{X}_{122}=2\begin{vmatrix}\beta', & \gamma', & \delta',\\ v, & w, & k,\\ q_{12}, & r_{12}, & s_{12},\end{vmatrix}+\begin{vmatrix}\beta, & \gamma, & \delta,\\ v, & w, & k,\\ q_{22}, & r_{22}, & s_{22},\end{vmatrix} \quad \mathrm{Y}_{122}=2\begin{vmatrix}\gamma', & \alpha', & \delta',\\ w, & u, & k,\\ r_{12}, & p_{12}, & s_{12},\end{vmatrix}+\begin{vmatrix}\gamma, & \alpha, & \delta,\\ w, & u, & k,\\ r_{22}, & p_{22}, & s_{22},\end{vmatrix} \quad \mathrm{Z}_{122}=\ldots, \ \mathrm{T}_{122}=\ldots$$

$$\mathrm{X}_{222}=\begin{vmatrix}\beta', & \gamma', & \delta',\\ v, & w, & k,\\ q_{22}, & r_{22}, & s_{22},\end{vmatrix} \qquad \mathrm{Y}_{222}=\begin{vmatrix}\gamma', & \alpha', & \delta',\\ w, & u, & k,\\ r_{22}, & p_{22}, & s_{22},\end{vmatrix} \qquad \mathrm{Z}_{222}=\ldots, \ \mathrm{T}_{222}=$$

This being so, let

$$\left.\begin{array}{l}\begin{vmatrix}\alpha, & \beta, & \gamma, & \delta, \\ \alpha', & \beta', & \gamma', & \delta', \\ u, & v, & w, & k, \\ p_{11}, & q_{11}, & r_{11}, & s_{11},\end{vmatrix}=\Upsilon, \quad \begin{vmatrix}\alpha, & \beta, & \gamma, & \delta, \\ \alpha', & \beta', & \gamma', & \delta', \\ u, & v, & w, & k, \\ p_{12}, & q_{12}, & r_{12}, & s_{12},\end{vmatrix}=\Upsilon_1, \quad \begin{vmatrix}\alpha, & \beta, & \gamma, & \delta, \\ \alpha', & \beta', & \gamma', & \delta', \\ u, & v, & w, & k, \\ p_{22}, & q_{22}, & r_{22}, & s_{22},\end{vmatrix}=\Upsilon_2, \\ \begin{vmatrix}\alpha, & \beta, & \gamma, & \delta, \\ u, & v, & w, & k, \\ p_{12}, & q_{12}, & r_{12}, & s_{12}, \\ p_{22}, & q_{22}, & r_{22}, & s_{22},\end{vmatrix}=\Omega_2, \quad \begin{vmatrix}\alpha, & \beta, & \gamma, & \delta, \\ u, & v, & w, & k, \\ p_{22}, & q_{22}, & r_{22}, & s_{22}, \\ p_{11}, & q_{11}, & r_{11}, & s_{11},\end{vmatrix}=-2\Omega_1, \quad \begin{vmatrix}\alpha, & \beta, & \gamma, & \delta, \\ u, & v, & w, & k, \\ p_{11}, & q_{11}, & r_{11}, & s_{11}, \\ p_{12}, & q_{12}, & r_{12}, & s_{12},\end{vmatrix}=\Omega.\end{array}\right\} \quad (68)$$

Also let Ω'_2, Ω'_1, Ω' represent the expressions obtained by writing α', β', γ', δ' for α, β, γ, δ respectively in the first lines of Ω_2, Ω_1, Ω. Then it will be found, by the method of compound determinants, that

$$\begin{vmatrix}X_{111}, & Y_{111}, \\ X_{112}, & Y_{112},\end{vmatrix}=2\Omega\begin{vmatrix}\gamma, & \delta, \\ w, & k,\end{vmatrix}+\Upsilon\begin{vmatrix}w, & k, \\ r_{11}, & s_{11},\end{vmatrix}=\begin{vmatrix}\gamma, & \delta, & \Upsilon, \\ w, & k, & . \\ r_{11}, & s_{11}, & 2\Omega,\end{vmatrix}$$

whence

$$\begin{vmatrix}X_{111}, & Y_{111}, & Z_{111}, \\ X_{112}, & Y_{112}, & Z_{112}, \\ X_{122}, & Y_{122}, & Z_{122},\end{vmatrix}=\begin{vmatrix}2\Upsilon, & \delta, & \Upsilon, \\ . & k, & . \\ 2\Omega'+2\Omega_1, & s_{11}, & 2\Omega,\end{vmatrix}=2k\{2\Upsilon_1\Omega-\Upsilon(\Omega_1+\Omega')\};$$

similarly,

$$\begin{vmatrix}X_{111}, & Y_{111}, & Z_{111}, \\ X_{112}, & Y_{112}, & Z_{112}, \\ X_{222}, & Y_{222}, & Z_{222},\end{vmatrix}=2k(\Upsilon_2\Omega-\Upsilon\Omega'_1);$$

both of which, when equated to zero, are comprised in the system

$$\frac{\Omega}{\Upsilon}=\frac{\Omega'+\Omega_1}{2\Upsilon_1}=\frac{\Omega'_1}{\Upsilon_2}. \quad \ldots \ldots \quad (69)$$

Similarly, taking the determinants

$$\begin{vmatrix}X_{112}, & Y_{112}, & Z_{112}, \\ X_{122}, & Y_{122}, & Z_{122}, \\ X_{222}, & Y_{222}, & Z_{222},\end{vmatrix}=0, \quad \begin{vmatrix}X_{111}, & Y_{111}, & Z_{111}, \\ X_{122}, & Y_{122}, & Z_{122}, \\ X_{222}, & Y_{222}, & Z_{222},\end{vmatrix}=0,$$

we should arrive at the system

$$\frac{\Omega_1}{\Upsilon}=\frac{\Omega'_1+\Omega_2}{2\Upsilon_1}=\frac{\Omega'_2}{\Upsilon_2}. \quad \ldots \ldots \quad (70)$$